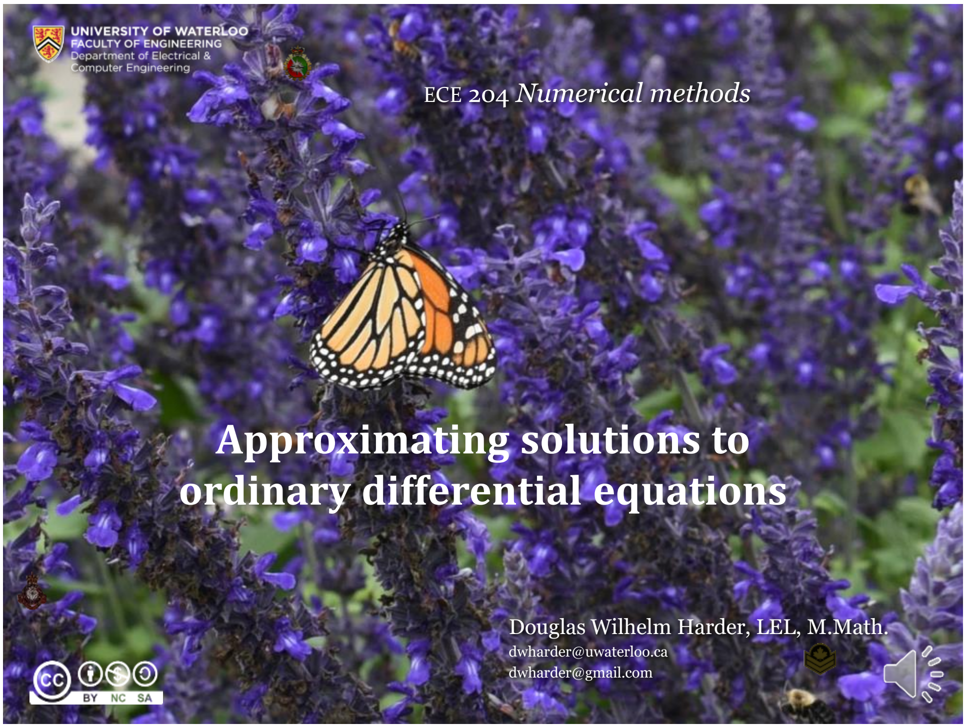


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# Introduction

- In this topic, we will
  - Describe what will be covered in this sub-topic
    - Approximating solutions to ordinary differential equations
  - Introduce some of the terminology
  - Describe the use of the implicit function theorem
  - Discuss the constraints necessary
  - Describe the upcoming lectures



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# Ordinary differential equations

- An ordinary differential equation (ODE) is any equation involving:
  - Algebraic expressions in $t$, an unknown function $y(t)$ and one or more of its derivatives
  - The *order* of an ODE is the highest derivative
  - This includes equations such as:

$$(y(t) + y^{(1)}(t) + \sin(y(t)) + \cos(y^{(1)}(t)))^2 = 1$$

- We will, however, focus on ODEs of the form:

$$y^{(1)}(t) = f(t, y(t))$$
$$y^{(2)}(t) = f(t, y(t), y^{(1)}(t))$$
$$y^{(3)}(t) = f(t, y(t), y^{(1)}(t), y^{(2)}(t))$$

  - That is, the highest derivative is a function of $t$, $y(t)$ and lower derivatives





Approximating solutions to analytic equations

# Systems of ODEs

- We will also include looking at systems of $n$ ordinary differential equations
  - $n$ equations in $t$, $n$ unknown functions $y_1(t), \ldots, y_n(t)$ and at least one derivative of each
  - One such example are the Lotka-Voltarra equations:

$$y_1^{(1)}(t) = \alpha y_1(t) - \beta y_1(t) y_2(t)$$
$$y_2^{(1)}(t) = -\gamma y_2(t) + \delta y_1(t) y_2(t)$$

  - These model:
    - Autocatalytic chemical reactions
    - Predator-prey relations

# Implicit function theorem

- We can make this assumption because of the implicit function theorem:

Theorem

Any equation of the form $F(x_0, x_1, \ldots, x_n) = 0$ can usually be written locally in the form $x_0 = f(x_1, \ldots, x_n)$

- For example, $t^2 + y^2(t) + \left[y^{(1)}(t)\right]^2 - 1 = 0$ can be written locally as one of

$$y^{(1)}(t) = \pm\sqrt{1 - t^2 - y^2(t)}$$

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# Constraints

- Recall that in order to find an interpolating polynomial of degree $n$,
  - We required $n + 1$ constraints, one for each coefficient
- An $n^{\text{th}}$ order differential equation requires that the $n^{\text{th}}$ derivative be integrated $n$ times
  - Each anti-derivative introduces one more unknown constant

$$y^{(3)}(t) = \cos(t)$$
$$y^{(2)}(t) = \sin(t) + c_0$$
$$y^{(1)}(t) = -\cos(t) + c_0 t + c_1$$
$$y(t) = -\sin(t) + \tfrac{1}{2}c_0 t^2 + c_1 t + c_2$$

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## Constraints

- Thus, an $n^{\text{th}}$ order differential equation requires *n* constraints
  - One possibility is to describe the state of the system perfectly at one moment in time:

$$y(t_0) = y_0$$
$$y^{(1)}(t_0) = y_0^{(1)}$$

    - This is called an initial condition, as it gives the state of the system at time $t_0$
  - Another approach is to describe the value of the function at different positions

$$u(a) = u_a$$
$$u(b) = u_b$$

    - This are called boundary conditions, they give the state of the system at the boundary of $[a, b]$







## Linear and non-linear equations

- In this first sub-section, we will look at approximating solutions to
  - Initial-value problems
  - Boundary-value problems

# Summary

- Following this topic, you now
  - Have reviewed concepts from your course in calculus
  - Understand the definition of an ordinary differential equation
  - Are aware of the implicit function theorem
  - Have considered the number of necessary constraints
  - Understand the constraints as being either
    - Initial values
    - Boundary values

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# References

[1] https://en.wikipedia.org/wiki/Ordinary_differential_equation

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# Acknowledgments

None so far.







# Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.
Mathematical equations are prepared in MathType by Design Science, Inc.
Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

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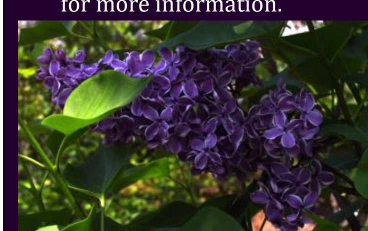

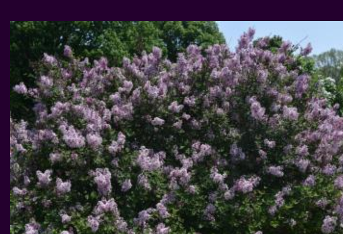

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